An anytime proof of the Brouwer fixed point theorem

Nisan Stiennon

Theiss Research

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Abstract

The usual elementary proof of Brouwer's fixed point theorem via Sperner's lemma is ineffective — it constructs a sequence of points that converges arbitrarily slowly to a fixed point. In this talk I will present a modified proof that uses Sperner multicolorings to produce bounds that monotonically converge to the set of fixed points.

Brouwer's fixed-point theorem is a central theorem of agent foundations. It's used to prove the existence of Nash equilibria, the existence of reflective oracles, the existence of logical inductors, etc.

It says that given a continuous function f from a simplex to itself, there is a fixed point x such that $f(x) = x$.

The elementary way to prove this is:

1. Derive a coloring of the simplex from f.
2. Triangulate the simplex.
3. Restrict the coloring to the vertices of the triangulation.
4. Sperner's lemma then says there is a little n-face whose vertices are all different colors.
5. By taking ever finer triangulations, we obtain a sequence of fully-colored faces. By the Bolzano-Weierstrass theorem, we can extract a convergent subsequence.
6. The limit must be a fixed point.

Step 5 is the unsatisfying one. It's ineffective in that it doesn't tell us how to find a convergent subsequence. You might naively expect that we could recursively triangulate the fully-colored face from step 4; but that doesn't work because it doesn't necessarily satisfy the hypotheses of Sperner's lemma. And the reason for that is that step 3 throws away a lot of information about f; if the triangulation is too coarse, we fail to capture any information about where the fixed points are.

The right way to prove the theorem, in my opinion, is to color all the faces, not just the vertices; and to allow multiple colors per face. Such a multicoloring retains enough information about f to approximate the fixed points with progressively tighter bounds — an anytime algorithm. The key is a combinatorial version of the index of a fixed point introduced by Bekker and Netsvetaev in 1995 [1].

I would be surprised if this proof is novel. But I've never seen it before.

Recall:

Definition 1. An *abstract simplicial complex* X is a family of *faces* which are nonempty subsets of a finite set of *vertices* V, such that if $\sigma \in X$ is a face, then X contains all nonempty subsets of σ. We define the *dimension* of a face to be $\dim(\sigma) = |\sigma| - 1$. In case $\dim(\sigma) = n$, we call σ an *n-face* and sometimes write it σ^n. We define $\dim(X) = \max_{\sigma \in X} \dim(\sigma)$. In case $\dim(X) = n$, we sometimes write X^n.

Definition 2. An *n-complex* X is an abstract simplicial complex where every face is a subset of an n-face. We sometimes write it X^n.

The *boundary* ∂X of an n-complex is the biggest $(n-1)$-subcomplex such that each $(n-1)$-face of ∂X is a subset of an odd number of n-faces in X.

Proposition 3. *The boundary of the boundary is empty: $\partial^2 X = \emptyset$*

Proof. Suppose X is an n-complex and $\tau^{n-2} \in \partial X$ is a face in the boundary. It is in $\partial^2 X$ just in case the following quantity is equal to $1 \pmod 2$:

$$\begin{aligned}
|\{\sigma^{n-1} \in \partial X | \tau \subset \sigma\}| &= \sum_{\substack{\sigma^{n-1} \\ \tau \subset \sigma}} [\sigma \in \partial X] \\
&= \sum_{\substack{\sigma^{n-1} \\ \tau \subset \sigma}} \sum_{\substack{\rho^{n} \\ \sigma \subset \rho}} 1 \\
&= \sum_{\substack{\rho^{n} \\ \tau \subset \rho}} \sum_{\substack{\sigma^{n-1} \\ \tau \subset \sigma \subset \rho}} 1 \\
&= 0
\end{aligned}$$

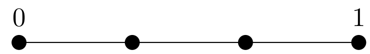
because there are exactly two $(n-1)$-simplices sandwiched between τ^{n-2} and ρ^n. ∎

Definition 4. An S-multicoloring of an abstract simplicial complex X is a map $\kappa : X \to \mathcal{P}(S)$ that assigns a nonempty subset of S to every face of X monotonically; that is,

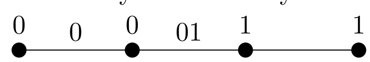
$$\tau \subset \sigma \quad \Rightarrow \quad \kappa(\tau) \subset \kappa(\sigma)$$

We'll always take S to be $\underline{n} = \{0, \ldots, n\}$ for some natural number n.

Example 5.



Given just the coloring on the boundary, we can conclude that there must be an edge that is colored both 0 and 1. For as we go from left to right, at some point the vertices must transition from being colored 0 to being colored not just 0, and then by monotonicity that edge must be colored both 0 and 1.



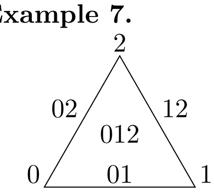
So we see that the coloring on the boundary tells us something about the number of fully-colored faces in the interior. The index makes this precise.

Now we introduce the Bekker-Netsvetaev index, based on [1], which defined it for single-colorings:

Definition 6. Given a multicolored abstract simplicial complex Z, an *index* on Z assigns to each subcomplex $X \subset Z$ an element $\mathrm{ind}(X) \in \mathbb{Z}/2$ satisfying:

1. If $\kappa(\sigma^0) = \underline{0}$, then $\mathrm{ind}(\sigma) = 1$.
2. If $\dim(X^n \cap Y^n) < n$, then $\mathrm{ind}(X \cup Y) = \mathrm{ind}(X) + \mathrm{ind}(Y)$.
3. If $n > 0$ and $\kappa(X^n) \subset \underline{n}$, then $\mathrm{ind}(X) = \mathrm{ind}(\partial X)$.
4. $\mathrm{ind}(X^n) = 1 \Rightarrow \kappa(X) \supset \underline{n}$.

Example 7.



This has index 1.

Theorem 8. *Every multicolored abstract simplicial complex has an index.*

Proof. For each face σ^n, define

$$\mathrm{ind}(\sigma^n) = \begin{cases} 1 & \text{if } n = 0 \text{ and } \kappa(\sigma) = \underline{0} \\ \mathrm{ind}(\partial\sigma) & \text{if } n > 0 \text{ and } \kappa(\sigma) \subset \underline{n} \\ 0 & \text{if } \kappa(\sigma) \not\subset \underline{n} \text{ and } \kappa(\sigma) \not\supset \underline{n} \\ \text{anything} & \text{if } \kappa(\sigma) \supsetneq \underline{n} \end{cases}$$

For each n-complex X, define

$$\mathrm{ind}(X) = \sum_{\sigma^n \in X} \mathrm{ind}(\sigma)$$

This clearly satisfies every part of the definition of index except axiom (4), which we will now prove.

If X is an n-complex and $\operatorname{ind}(X) = 1$, then for some $\rho^n \in X$ we must have $\operatorname{ind}(\rho^n) = 1$. If we knew that $\kappa(\rho^n) \supset \underline{n}$, then by monotonicity we'd be done. So it suffices to show axiom (4) for simplices.

Suppose σ^n is an n-face.

The case $n = 0$ is clear. So suppose $n > 1$.

If $\kappa(\sigma) \supset \underline{n}$, then axiom (4) is satisfied. So suppose $\kappa(\sigma) \not\supset \underline{n}$.

If $\kappa(\sigma) \not\subset \underline{n}$, then $\operatorname{ind}(\sigma) = 0$ and axiom (4) is vacuously true. So suppose $\kappa(\sigma) \subset \underline{n}$.

Our assumptions so far imply $\kappa(\sigma) \subsetneq \underline{n}$.

If $\kappa(\sigma) \not\supset \underline{n-1}$, then $\operatorname{ind}(\sigma) = \operatorname{ind}(\partial\sigma) = 0$ by induction. So suppose $\kappa(\sigma) \supset \underline{n-1}$.

Our assumptions so far imply $\kappa(\sigma) = \underline{n-1}$.

If $n = 1$, then $\operatorname{ind}(\sigma) = \operatorname{ind}(\partial\sigma)$, and $\partial\sigma$ is two vertices colored $\underline{0}$, which has index 0. So assume $n > 1$.

So $\operatorname{ind}(\sigma) = \operatorname{ind}(\partial\sigma) = \operatorname{ind}(\partial^2\sigma) = \operatorname{ind}(\emptyset) = 0$. □

The index has a unique value on n-complexes with well-behaved boundary:

Theorem 9. *Suppose X is an $\underline{n}$-multicolored n-complex, and every face $\sigma^{n-1} \in \partial X$ satisfies $\kappa(\sigma) \neq \underline{n}$. Then $\operatorname{ind}(X)$ is uniquely determined.*

Proof. If $n = 0$, we have $\operatorname{ind}(X) = |X| \pmod 2$. We proceed by induction on n.

Let Y^{n-1} be the biggest $\underline{n-1}$-multicolored $(n-1)$-subcomplex of ∂X. Then

$$\operatorname{ind}(X) = \operatorname{ind}(\partial X) = \operatorname{ind}(Y) + \sum_{\sigma^{n-1} \in \partial X \setminus Y} \operatorname{ind}(\sigma)$$

The sum vanishes. Indeed, $\kappa(\sigma) \neq \underline{n}$ because $\sigma \in \partial X$ and $\kappa(\sigma) \not\subset \underline{n-1}$ because $\sigma \notin Y$. Thus $\kappa(\sigma) \not\supset \underline{n-1}$, so $\operatorname{ind}(\sigma) = 0$.

Thus $\operatorname{ind}(X) = \operatorname{ind}(Y)$. We claim that $\kappa(\tau) \neq \underline{n-1}$ for all $\tau^{n-2} \in \partial Y$. Then by the inductive hypothesis, $\operatorname{ind}(Y)$ is uniquely determined.

To prove the claim, let $\tau^{n-2} \in \partial Y$. There are an odd number of $\sigma^{n-1} \in Y$ such that $\tau \subset \sigma$. And $\tau \notin \partial^2 X = \emptyset$, so there are an even number of $\sigma^{n-1} \in \partial X$ such that $\tau \subset \sigma$. So let $\sigma^{n-1} \in \partial X \setminus Y$ and $\tau \subset \sigma$.

By hypothesis, $\kappa(\sigma) \neq \underline{n}$. And by the definition of Y, we have $\kappa(\sigma) \not\subset \underline{n-1}$. Therefore $\kappa(\sigma) \not\supset \underline{n-1}$. By monotonicity $\kappa(\tau) \not\supset \underline{n-1}$, which proves the claim. □

Going forward, whenever the hypotheses of Theorem 9 hold, we'll refer to the value $\operatorname{ind}(X)$ as "the index of X".

Definition 10. Suppose X is the abstract simplicial complex corresponding to a subdivision of the n-simplex Δ^n. For each $V \subset \underline{n}$, there is a $(|V| - 1)$-subcomplex $X_V \subset X$ corresponding to a subdivision of the face of Δ^n with vertices V.

A multicoloring κ of X is a *Sperner multicoloring* if $\kappa(X_V) \subset V$ for every $V \subset \underline{n}$.

Now for a version of Sperner's lemma for multicolorings:

Lemma 11 (Sperner's lemma)**.** *If X is a subdivision of Δ^n and κ is a Sperner multicoloring on it, then* $\mathrm{ind}(X) = 1$.

Proof. For $n = 0$, X is a point colored $\underline{0}$, so it has index 1.

For $n > 0$,

$$\mathrm{ind}(X) = \mathrm{ind}(\partial X) = \sum_{0 \le v < n} \mathrm{ind}(X_{\underline{n}\backslash\{v\}}) + \mathrm{ind}(X_{\underline{n-1}})$$

By induction, the $\mathrm{ind}(X_{\underline{n-1}})$ term is equal to 1. The other terms are zero because $\kappa(X_{\underline{n}\backslash\{v\}}) \subset \underline{n}\backslash\{v\}$ and so $\kappa(X_{\underline{n}\backslash\{v\}}) \not\supset \underline{n-1}$. ☐

Now we're ready for the more-effective proof of Brouwer's fixed point theorem.

Theorem 12 (Brouwer fixed-point theorem)**.** *If $f : \Delta^n \to \Delta^n$ is a continuous map on a geometric simplex, it has a fixed point.*

Proof. Define the difference multicoloring:

$$\begin{aligned} \delta : \Delta^n &\to \underline{n} \\ [x_0 : \cdots : x_n] &\mapsto \{k \in \underline{n} \mid x_k \ge f(x)_k\} \end{aligned}$$

Note that if $\delta(x) = \underline{n}$, then $f(x) = x$.

Also, $\delta(x)$ is always nonempty.

Also, it is closed: If $k \in \delta(x^i)$ for every x^i, then $k \in \delta(\lim_i x^i)$.

Also, if x belongs to the face with vertices $V \subset \underline{n}$, then $x_k = 0$ for each $k \notin V$. Then $x_k > 0$ for some $k \in \delta(x)$. So $\delta(x) \cap V$ must be nonempty.

Thus $\kappa(x) := \delta(x)\backslash\{v \in \underline{n} \mid x_v = 0\}$ is always nonempty and is a Sperner multicoloring. It is not closed, though.

Take the i-fold barycentric subdivision of Δ^n, and let X_i denote its associated abstract simplicial complex, so that $|X_i| = \Delta^n$. For each face $\sigma \in X_i$, let $\kappa(\sigma) := \cup_{x \in |\sigma|} \kappa(x)$. This makes X_i into a Sperner-multicolored n-complex.

By Sperner's lemma (Lemma 11), the index of X_i is 1. Thus X_i contains at least one n-face σ^n such that $\kappa(\sigma) = \underline{n}$. Let Y_i be the biggest n-complex all of whose n-faces satisfy $\kappa(\sigma^n) = \underline{n}$.

We have a nested sequence $|Y_0| \supset |Y_1| \supset \cdots$. Since they are all compact and nonempty, $Y = \cap |Y_i|$ is also nonempty.

I claim that Y is the set of points fixed by f. Indeed, suppose $y \in Y$ and $0 \le v \le n$. For every i, we have $y \in |\sigma^n|$ for some $\sigma^n \in Y_i$. Since $\kappa(\sigma) = \underline{n}$, there exists some $x_i \in |\sigma|$ with $\kappa(x_i) \ni v$. Thus $\delta(x_i) \ni v$. Iterated barycentric subdivisions shrinks the diameter of faces to zero, so $x_i \to y$. By closedness, $\delta(y) \ni v$. Since this holds for all v, we have $\delta(y) = \underline{n}$, and so y is a fixed point.

(Conversely, if y is a fixed point then every face it belongs to is fully-colored.)

☐

The sequence $|Y_0| \supset |Y_1| \supset \cdots$ eventually excludes every non-fixed point.

However, if $y \in Y$ there's no way to know this. So it's still not constructive.

You only really need to apply Sperner's Lemma at the first stage, because subdividing preserves index. See Appendix.

A Appendix

A variant of the proof of Theorem 12 uses the fact that subdividing Y_i does not affect the index.

Definition 13. A *quasi-subdivision* of an S-multicolored n-complex (X, κ) is an S-multicolored n-complex (Y, λ) together with a map $p : Y \to X$ that maps every face of Y to a face of X such that:

1. $\tau \subset \sigma \Rightarrow p(\tau) \subset p(\sigma)$
2. $\dim(p(\sigma)) \geq \dim(\sigma)$
3. p is surjective
4. If Z is a k-subcomplex of X, then $p^{-1}(Z)$ is a k-subcomplex of Y
5. If Z is a k-subcomplex of X, then $p^{-1}(\partial Z) = \partial p^{-1}(Z)$
6. $\kappa(\sigma) = \lambda(p^{-1}(\sigma))$

Proposition 14. *Given a quasi-subdivision p from (Y, λ) to (X, κ) and an index on Y, the formula* $\mathrm{ind}(Z) = \mathrm{ind}(p^{-1}(Z))$ *defines an index on X.*

Proof. The proof is straightforward.

1. If $\kappa(\sigma^0) = \underline{0}$, then by (2) and (6) we have $\mathrm{ind}(\sigma) = 1$.
2. If Z is a k-subcomplex of X and $\mathrm{ind}(Z) = 1$, then by (4) and (6), we have $\kappa(Z) \supset \underline{k}$.
3. If Z, W and k-subcomplexes of X with no k-faces in common, then by (4) $p^{-1}(Z)$ and $p^{-1}(W)$ are k-subcomplexes of Y, and by (2) they have no k-faces in common. Thus $\mathrm{ind}(Z \cup W) = \mathrm{ind}(Z) \cup \mathrm{ind}(W)$.
4. If Z is a k-complex with $k > 0$ and $\kappa(Z) \subset \underline{n}$, then by (4), (6), and (5), we have $\mathrm{ind}(Z) = \mathrm{ind}(\partial Z)$

□

References

[1] Boris M. Bekker and Nikita Yu. Netsvetaev. "Generalized Sperner lemma and subdivisions into simplices of equal volume". In: *Zap. Nauchn. Sem. POMI* 231 (1995), pp. 245–254. URL: `https://www.mathnet.ru/eng/znsl3754`.